ARCS AND RESOLUTION OF SINGULARITIES

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Abstract. For a certain class of varieties X, we derive a formula for the valuation d_X on the arc space $\mathcal{L}(Y)$ of a smooth ambient space Y, in terms of an embedded resolution of singularities. A simple transformation rule yields a formula for the geometric Poincaré series. Furthermore, we prove that for this class of varieties, the arithmetic and the geometric Poincaré series coincide. We also study the geometric valuation for plane curves.

1. Introduction

Let k be a field of characteristic zero, and let k^{alg} be an algebraic closure. Let X be a subvariety of affine space $\mathbb{A}^d_k$, defined by polynomial equations $f_j(x) = 0$, $j = 1, \ldots, r$.

We can describe jets on X in terms of the coordinate system $(x_1, \ldots, x_d)$ on $\mathbb{A}^d_k$. An n-jet on X is a tuple of truncated power series

$$a = (a_{1,0} + a_{1,1}t + \ldots + a_{1,n}t^n, \ldots, a_{d,0} + a_{d,1}t + \ldots + a_{d,n}t^n)$$

with coefficients in k^{alg}, such that $f_j(a) = 0 \, mod \, t^{n+1}$ for each j. Jets can be considered as approximate solutions for the system $f_j = 0$. Using the $a_{k,l}$ as affine coordinates, we give the set of n-jets the structure of a subvariety $\mathcal{L}_n(X)$ of $\mathbb{A}^{d(n+1)}_k$. There are obvious truncation maps

$$\pi^m_n : \mathcal{L}_m(X) \to \mathcal{L}_n(X)$$

for $m \geq n$. Similarly, the set of arcs on X, that is, d-tuples ψ of power series over k^{alg} satisfying $f_j(\psi) = 0$ for each j, can be seen as the set of closed points of a k-scheme $\mathcal{L}(X)$, which comes with truncation maps

$$\pi_n : \mathcal{L}(X) \to \mathcal{L}_n(X)\,.$$

Exact constructions and definitions are given in the next section.

We can attach three motivic generating series to the variety X. The Igusa Poincaré series $Q(T)$ counts all n-jets in $\mathcal{L}_n(X)$, using the universal Euler characteristic, taking values in the Grothendieck group of varieties over k. The geometric Poincaré series $P_{geom}(T)$ only takes n-jets into account which can be lifted through π_n to an arc on X. Finally, the arithmetic Poincaré series P_{arith} counts, for each field K containing k, the K-rational n-jets that can be lifted to a K-rational point of $\mathcal{L}(X)$. All three series are rational (over the appropriate coefficient rings), see the work of Denef and Loeser, in particular the survey article [4].

While the Igusa Poincaré series can easily be expressed in terms of a resolution of singularities, the geometric and arithmetic series are very hard to compute in general. The proof of their rationality is a qualitative proof, using results from

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model theory like quantifier elimination, and does not yield quantitative results. Up to now, the series had only been computed for analytic branches of plane curves, and for toric surfaces [7][17][18], using Puiseux pairs and a specific representation of arcs on toric varieties. In this paper, we present a formula for both series in terms of a resolution of singularities satisfying certain conditions, provided that such a resolution exists. In fact, we will prove that in this case, the series coincide. This opens a whole new realm of varieties X for which the series can easily be computed, including the toric surfaces. In particular, our results imply the rationality of the geometric and arithmetic series. Furthermore, our methods leave much room for generalization, unlike the methods used to compute the cases mentioned above. We believe that, at least in theory, you can use similar arguments to compute the geometric Poincaré series of any variety, the only restriction being the combinatorial complexity.

To be precise, we determine the maximal truncation of an arc ψ in smooth ambient space that can be lifted to an arc on X, and we construct an optimal approximation in $\mathcal{L}(X)$, all in terms of the exact location of the lifting of ψ through the resolution morphism on the exceptional locus.

Sections 2 and 3 contain some preliminaries on jets and motivic integrals, and section 4 deals with the plane curve case. In section 5, the general formulae for P_{geom} are established. Section 6 partially answers a question from [17], concerning quasirational singularities. In section 7, we give a very short computation of the geometric series of a toric surface. Finally, in Section 8, we discuss the arithmetic series.

2. Motivic integration and the geometric Poincaré series

Until further notice, k is an algebraically closed field of characteristic zero.

Let X be a variety over k, that is, a reduced and separated scheme of finite type over k, not necessarily irreducible. For each positive integer n, the functor from the category of k-algebras to the category of sets, sending an algebra R to the set of $R[[t]]/t^{n+1}R[[t]]$-rational points on X, is representable by a variety $\mathcal{L}_n(X)$. Since the natural projections $\pi_n^{n+1}:\mathcal{L}_{n+1}(X)\rightarrow\mathcal{L}_n(X)$ are affine, we can take the projective limit in the category of schemes to obtain the scheme of arcs $\mathcal{L}(X)$. This scheme represents the functor sending a k-algebra R to the set of $R[[t]]$-rational points on X, and comes with natural projections $\pi_n:\mathcal{L}(X)\rightarrow\mathcal{L}_n(X)$, mapping an arc to its n-truncation. For a subvariety Z of X, we define $\mathcal{L}(X)_Z$ to be the closed subscheme $\pi_0^{-1}(Z)$ of $\mathcal{L}(X)$. When X is smooth, the morphisms π_n^{n+1} are Zariski-locally trivial fibrations with fiber $\mathbb{A}^d_k$, where d is the dimension of X. A morphism h from X to Y induces a morphism h from $\mathcal{L}(X)$ to $\mathcal{L}(Y)$ by composition.

We now introduce the Grothendieck ring $K_0(Var_k)$ of varieties over k. Start from the free abelian group generated by isomorphism classes $[X]$ of varieties X over k, and consider the quotient by the relations $[X]=[X\setminus X']+[X']$, where X' is closed in X. A constructible subset of X can be written as a disjoint union of locally closed subsets and determines unambiguously an element of $K_0(Var_k)$. The Cartesian product induces a product on $K_0(Var_k)$, which makes it a ring. We denote the class of the affine line $\mathbb{A}^1_k$ in $K_0(Var_k)$ by $\mathbb{L}$, and the localization of $K_0(Var_k)$ with respect to $\mathbb{L}$ by $\mathcal{M}_k$. On $\mathcal{M}_k$, we consider a decreasing filtration F^m, where F^m is the subgroup generated by elements of the form $[X]\mathbb{L}^{-i}$, with

$\dim X - i \leq -m$. We define $\hat{\mathcal{M}}_k$ to be the completion of $\mathcal{M}_k$ with respect to this filtration.

The geometric Poincaré series, a formal power series over $K_0(Var_k)$, is defined to be

$$P_{geom}(T) = \sum_{n \geq 0} [\pi_n(\mathcal{L}(X))] T^n \; .$$

Denef and Loeser [5] proved that it is rational in $\mathcal{M}_k[[T]]$. The series is well defined, since Greenberg's theorem [13] states that we can find a positive integer c such that, for all n, and for each field K containing k, $\pi_n(\mathcal{L}(X)(K)) = \pi_n^{nc}(\mathcal{L}_{nc}(X)(K))$. So it follows from Chevalley's theorem [15] that $\pi_n(\mathcal{L}(X))$ is constructible, and hence determines an element $[\pi_n(\mathcal{L}(X))]$ in $K_0(Var_k)$. One can define local variants of this series, e.g. by only considering arcs with origin in a fixed point x of X.

Let $X \subset Y$ be varieties over k, with Y smooth and of dimension d. We define a valuation d_X on $\mathcal{L}(Y)$ as follows: $d_X(\psi) = s$ if $\pi_{s-1}(\psi) \in \pi_{s-1}\mathcal{L}(X)$, but $\pi_s(\psi) \notin \pi_s\mathcal{L}(X)$, where we consider $\mathcal{L}(X)$ as a subspace of $\mathcal{L}(Y)$. We define $d_X(\psi)$ to be ∞ when $\psi \in \mathcal{L}(X)$, and to be 0 if $\psi(0) \notin X$. When X is smooth, ψ is a k-rational arc on Y with origin at x, and $\mathcal{I}$ is the defining ideal sheaf of X in Y,

$$d_X(\psi) = ord_t\, \mathcal{I}(\psi) := \min\{ord_t\, f(\psi) \mid f \in \mathcal{I}_x\} \, .$$

In general, we will call $ord_t\, \mathcal{I}(\psi)$ the order of contact between ψ and X, and we will denote this by $c(\psi, X)$.

For each positive integer s, and each point x on X, we define $D_x(X, s)$ by the motivic integral

$$D_x(X, s) = \int_{\mathcal{L}(Y)_x} \mathbb{L}^{-d_X(\psi)s} d\mu(\psi) \, .$$

We refer the reader to [5][6][9][18] for an introduction to motivic integration. The normalization of the motivic measure we use is the same as in these articles.

When it is clear which point x and which variety X the integral $D_x(X, s)$ is associated to, we omit the subscript x and the variable X from our notation. Observe that $D_x(X, s)$ also depends on the ambient space Y. Putting $\mathbb{L}^{-s}$ equal to T, the following simple formula relates the local geometric Poincaré series P_{geom} of X at x to $D_x(s)$.

Lemma 1.

$$P_{geom}(\mathbb{L}^{-d}T) = \frac{1 - \mathbb{L}^d D_x(T)}{1 - T} \text{ in } \hat{\mathcal{M}}_k[[T]] \, .$$

Proof. The general T^n-term of the right side can be written as

$$1 - \mathbb{L}^d \sum_{i=1}^{n} D_x(T)[i]$$

where $D_x(T)[i]$ denotes the coefficient of T^i in $D_x(T)$. Now it suffices to observe that $P_{geom}(\mathbb{L}^{-d}T)[n]$ equals $\mathbb{L}^d$ times the motivic measure of the arcs ψ in $\mathcal{L}(X)_x$ satisfying $d_X(\psi) > n$, while 1 is equal to $\mathbb{L}^d$ times the total measure of $\mathcal{L}(X)_x$, and $\sum_{i=1}^n D_x(T)[i]$ is the measure of the cylinder of arcs ψ satisfying $d_X(\psi) \leq n$. The lemma now follows from the additivity of μ. $\square$

3. Computing motivic integrals

This section contains some trivial remarks, concerning the computation of motivic integrals, using blow-ups and the change of variables formula.

Let $X_1, \ldots, X_t$ be smooth subvarieties of a smooth d-dimensional ambient space Y, intersecting transversally along a subvariety Z. Let c_i be the codimension of X_i in Y. We suppose that the sum of the c_i does not exceed the dimension d of Y. One can check immediately that

$$\int_{\mathcal{L}(Y)_Z} \mathbb{L}^{-\sum \alpha_i c(\psi, X_i)} d\mu(\psi) = [Z]\mathbb{L}^{-d} \prod_{i=1}^{t} (\mathbb{L}^{c_i} - 1) \frac{\mathbb{L}^{-\alpha_i - c_i}}{1 - \mathbb{L}^{-\alpha_i - c_i}}, \tag{1}$$

where the coefficients α_i are positive integers. In practice, one reduces to this situation using resolution of singularities and the change of variables formula for motivic integrals.

Now suppose that $t = 2$, and let A be the measurable subset of $\mathcal{L}(Y)$ consisting of all arcs ψ, satisfying $\psi(0) \in Z$ and $c(\psi, X_1) \geq c(\psi, X_2)$. We can easily compute the motivic integral

$$I = \int_A \mathbb{L}^{-\alpha_1 c(\psi, X_1) - \alpha_2 c(\psi, X_2)} d\mu(\psi)$$

by blowing up Z, and applying the change of variables formula. Let ψ be an arc on Y, with origin in Z but not entirely contained in Z, and let ψ' be its lifting through the blow-up $h : Y' \to Y$ with center Z and exceptional divisor E. Let X_i' be the strict transform of X_i. It is clear that ψ belongs to A if and only if $\psi'(0) \notin X_2'$. The change of variables formula yields

$$\begin{aligned} I &= \int_{\mathcal{L}(Y')_{E \setminus X_2'}} \mathbb{L}^{-\alpha_1 c(\psi', X_1') - (\alpha_1 + \alpha_2 + c_1 + c_2 - 1) c(\psi', E)} d\mu(\psi') \\ &= \mathbb{L}^{-d}(\mathbb{L} - 1)\{[E \setminus (X_1' \cup X_2')] \frac{\mathbb{L}^{-(\alpha_1 + \alpha_2 + c_1 + c_2)}}{1 - \mathbb{L}^{-(\alpha_1 + \alpha_2 + c_1 + c_2)}} \\ &\quad + [E \cap X_1'](\mathbb{L}^{c_1} - 1) \frac{\mathbb{L}^{-(2\alpha_1 + \alpha_2 + 2c_1 + c_2)}}{(1 - \mathbb{L}^{-(\alpha_1 + \alpha_2 + c_1 + c_2)})(1 - \mathbb{L}^{-\alpha_1 - c_1})}\}. \end{aligned}$$

Of course, analogous statements can be formulated for $t > 2$, or more complicated sets A.

As a final example, we compute the motivic integral

$$I = \int_{\mathcal{L}(Y)_Z} \mathbb{L}^{-\lfloor c(\psi, X_1)/2 + c(\psi, X_2)/2 \rfloor} d\mu(\psi),$$

where $\lfloor x \rfloor$ is the largest integer smaller than or equal to x. Blowing up Z, as before, we get

$$I = \int_{\mathcal{L}(Y')_E} \mathbb{L}^{-(c_1 + c_2) c(\psi', E) - \lfloor c(\psi', X_1')/2 + c(\psi', X_2')/2 \rfloor} d\mu(\psi').$$

The advantage of this method is that $X_1' \cap E$ and $X_2' \cap E$ are disjoint. A straightforward computation yields

$$\begin{aligned} I = \mathbb{L}^{-d}(\mathbb{L} - 1) \frac{\mathbb{L}^{-(c_1 + c_2 + 1)}}{1 - \mathbb{L}^{-(c_1 + c_2 + 1)}} &\{[E \setminus (X_1' \cup X_2')] \\ &+ [E \cap X_1'](\mathbb{L}^{c_1} - 1) \frac{\mathbb{L}^{-c_1} + \mathbb{L}^{-2c_1 - 1}}{1 - \mathbb{L}^{-2c_1 - 1}} + [E \cap X_2'](\mathbb{L}^{c_2} - 1) \frac{\mathbb{L}^{-c_2} + \mathbb{L}^{-2c_2 - 1}}{1 - \mathbb{L}^{-2c_2 - 1}}\}. \end{aligned}$$

These and similar methods will allow us to compute the geometric Poincaré series from the formula for d_X, that we will establish in a subsequent section. As horrifying as the computations may look, they are all based on the same basic principles: blowing up in order to obtain transversal intersection, and to simplify the integration domain, and applying the change of variables formula.

4. Plane curves

Let k be an algebraically closed field of characteristic 0. Let X be a formal branch of a plane algebraic curve over k, with Puiseux expansion

$$\begin{cases} x &= t^m \\ y &= \sum_p a_p t^p , \end{cases}$$

where we suppose $m < \min\{p \,|\, a_p \neq 0\}$. Let (p_i, q_i), $i = 1, \ldots, s$ be the characteristic pairs, and let $c_1 t^{k_1} + \ldots + c_s t^{k_s}$ be the corresponding essential terms in the expansion of y. As is explained in [3], the knowledge of the characteristic pairs suffices to construct the resolution graph of the minimal embedded resolution $h : \tilde{Y} \to Y = \mathbb{A}^2_k$ of X. We will derive a formula for the valuation function $d_X \circ h$ in terms of the contact of an arc on $\tilde{Y}$ with the exceptional locus of h, and with the strict transform $\tilde{X}$ of X.

Let us recall what the resolution graph of h looks like. In addition to the numerical data (N_j, ν_j), we attach to each irreducible exceptional divisor E_j a couple (M_j, μ_j). Let E_k, $k \in K(j)$, be the exceptional divisors containing the point that was blown up in the creation of E_j; thus $K(j)$ has at most two elements, and may be empty. We define the multiplicity M_j to be equal to N_j, subtracted by the sum of the N_k, $k \in K(j)$, and we put μ_j equal to the sum of the μ_k, where the divisor emerging in the first blow-up gets initial value $\mu = 1$.

For each index $i = 1, ..., s$, we get a chain P_i of exceptional divisors. In order to describe what P_i looks like, we first introduce some new invariants. Consider the following Euclidean algorithm:

$$\begin{aligned} \kappa_i &= a_{i,1} r_{i,1} + r_{i,2} \\ r_{i,1} &= a_{i,2} r_{i,2} + r_{i,3} \\ &\cdots \\ r_{i,w(i)-1} &= a_{i,w(i)} r_{i,w(i)} , \end{aligned}$$

where $\kappa_1 = k_1$, $\kappa_i = k_i - k_{i-1}$, $r_{1,1} = m$, and $r_{i,1} = r_{i-1,w(i-1)}$. Given $P_1, \ldots, P_{i-1}$, the resolution process runs as follows: first, we get a chain of divisors $E_{i,1,j}$, $j = 1, \ldots, a_{i,1}$, each with multiplicity $r_{i,1}$. Thereupon, $a_{i,2}$ divisors $E_{i,2,j}$ with multiplicity $r_{i,2}$ emerge, each of them separating the previous one from $E_{i,1,a_{i,1}}$. This process continues, so at the end, P_i is the chain starting with

$$E_{i,1,1}, \ldots, E_{i,1,a_{i,1}}, E_{i,3,1}, \ldots, E_{i,3,a_{i,3}}, \ldots$$

and ending in

$$\ldots, E_{i,4,a_{i,4}}, \ldots, E_{i,4,1}, E_{i,2,a_{i,2}}, \ldots, E_{i,2,1} \ .$$

Denote $E_{i,w(i),a_{i,w(i)}}$ by F_i. If $i < s$, the divisor $E_{i+1,1,1}$ intersects F_i in a smooth point of P_i, and $\tilde{X}$ intersects F_s in a smooth point of $\cup P_i$.

Let ψ be a non-constant k-rational arc on $\mathbb{A}^2_k$, with $\psi(0) = (0,0)$, and let $\tilde{\psi}$ be the unique lifting of ψ through h. By a lifting φ' of an arc φ on a variety Z through a proper birational morphism $f : W \to Z$, we mean the following: suppose that f

is an isomorphism over a Zariski-open neighbourhood in Z of the generic point of φ. Applying the valuative criterion for properness to the morphism f, we see that there exists a unique arc φ' on W such that the composition $h \circ \varphi'$ is equal to φ. We call this φ' the lifting of φ through f.

We will give a formula for $d_X(\psi)$ in terms of the contact of $\tilde{\psi}$ with the exceptional locus and $\tilde{X}$. This is possible because this contact information allows us to reconstruct the relevant part of the power series expansion of ψ.

If $x = \tilde{\psi}(0)$ is a smooth point of $\cup P_i$, contained in $E_{1,1,1}$, the constant arc at the origin of $\mathbb{A}^2_k$ is an optimal approximation for ψ in $\mathcal{L}(X)$ (with respect to the valuation ord_t), so $d_X(\psi)$ equals the multiplicity n of ψ. If not, we may assume $\psi(s)$ to be of the form

$$\left\{ \begin{array}{lcl} x & = & s^n \\ y & = & \sum_p b_p s^p \,, \end{array} \right.$$

with $n < \min\{p\,|b_p \neq 0\}$. In both cases, it is easily verified that the multiplicity n is equal to $\sum \mu_k \gamma_k$, where the sum is taken over the exceptional components E_k containing x, and γ_k is the order of contact of $\tilde{\psi}$ with E_k. If n is not a multiple of m, the constant arc at the origin will again be an optimal approximation for ψ, so $d_X(\psi) = n$. From now on, we suppose that $n = \lambda m$, $\lambda \in \mathbb{N}$.

First, suppose that $x \notin \tilde{X}$ is contained in $E_{i,j,k} \neq F_i$, with j even. This implies that ψ agrees with the Puiseux expansion of X, modulo a reparametrization $t = s^\lambda$, up to the essential term $c_i t^{k_i}$, so $d_X(\psi) = \lambda k_i$. If j is odd, $E_{i,j,k}$ is the only exceptional component containing x, and $x \notin \tilde{X}$, it is easy to see that

$$\lambda^{-1} d_X(\psi) = k_{i-1} + a_{i,1} r_{i,1} + a_{i,3} r_{i,3} + \ldots + k\, r_{i,j}\,,$$

where we put $k_0 = 0$. Next, assume that $x \notin \tilde{X}$ is the intersection point of $E_{i,j,k}$ and $E_{i,j,k+1}$, where j is odd. In this case,

$$\lambda^{-1} d_X(\psi) = k_{i-1} + a_{i,1} r_{i,1} + a_{i,3} r_{i,3} + \ldots + k\, r_{i,j} + \lambda^{-1}\gamma,$$

where γ is the order of contact of $\tilde{\psi}$ with $E_{i,j,k+1}$. If x is the intersection point of $E_{i,j,a_{i,j}}$ and $E_{i,j+2,1}$, where j is odd, then

$$\lambda^{-1} d_X(\psi) = k_{i-1} + a_{i,1} r_{i,1} + a_{i,3} r_{i,3} + \ldots + a_{i,j}\, r_{i,j} + \lambda^{-1}\gamma,$$

where γ is the order of contact of $\tilde{\psi}$ with $E_{i,j+2,1}$. And if x is the intersection point of $E_{i,j,a_{i,j}}$ and F_i, where j is odd, and $w(i)$ even,

$$\lambda^{-1} d_X(\psi) = k_{i-1} + a_{i,1} r_{i,1} + a_{i,3} r_{i,3} + \ldots + a_{i,j}\, r_{i,j} + \lambda^{-1}\gamma,$$

where γ is the order of contact of $\tilde{\psi}$ with F_i. Finally, if $x \in \tilde{X}$, $d_X(\psi)$ equals $\lambda k_s + \gamma$, where this time γ is the contact order of $\tilde{\psi}$ with $\tilde{X}$.

This analysis allows one to compute the geometric Poincaré series of Y - which was already computed in a much more elementary way in [7] - using the motivic change of variables formula

$$\int_{\mathcal{L}(Y)} \mathbb{L}^{-d_X} d\mu = \int_{\mathcal{L}(\tilde{Y})} \mathbb{L}^{-d_X \circ h - ord_t\, Jac_h} d\mu\,.$$

One can use the same techniques to compute the geometric Poincaré series for plane curves which are not necessarily analytically irreducible.

5. General results

Let $X \subset Y$ be varieties over an algebraically closed field k of characteristic zero, with Y smooth and of dimension d, and X of dimension m. Let $h : Y' \to Y$ be a composition $h_r \circ \cdots \circ h_1$, where each h_i is the blow-up of a point, and, if $i > 1$, this point lies on at most one exceptional divisor of $h_{i-1} \circ \cdots \circ h_1$. Let X' be the strict transform of X under h, and suppose that X' is smooth. Assume furthermore that X' and the exceptional locus E intersect transversally, and that each exceptional component E_i of E contains a point of X' that does not lie on any other exceptional component. Let ψ be k-rational arc in $\mathcal{L}(Y)_X \setminus \mathcal{L}(X)$, and let ψ' be the lifting of ψ through h. If $\psi'(0) \notin E$, put $\lambda = 0$. If $\psi'(0)$ lies on precisely one exceptional component E_i of E, we define λ to be equal to $c(\psi', E_i)$, multiplied by the order ν_i of the Jacobian of h on E_i, divided by $d-1$. This latter factor $\nu_i/(d-1)$ indicates the depth e_i of E_i in the composition of blow-ups. Finally, if $\psi'(0)$ lies on two exceptional components E_{i+1} and E_i, where E_{i+1} was created by blowing up a point of E_i, we put λ equal to

$$c(\psi', E_i)e_i + c(\psi', E_{i+1})e_{i+1} \, .$$

With this notation, we can formulate the following theorem.

Theorem 1. *Under the conditions explained above,*

$$d_X(\psi) = d_{X'}(\psi') + \lambda \, .$$

Proof. The fact that we are considering arcs, allows us to work locally with respect to the étale topology. Using our assumptions on h, the following lemma is easily verified.

Lemma 2. *We can find local coordinates $(y_1, \ldots, y_d)$ on Y at $\psi(0)$, and local coordinates $(y'_1, \ldots, y'_d)$ on Y' at $\psi'(0)$, such that the following properties are satisfied:*

- *If $\psi'(0)$ is contained in exactly one exceptional component E_i, the morphism h is given by*

  $$h(y'_1, \ldots, y'_d) = (y'_1, (y'_1)^{e_i} y'_2, \ldots, (y'_1)^{e_i} y'_m, (y'_1)^{e_i}\Phi_{m+1}, \ldots, (y'_1)^{e_i}\Phi_d) \, .$$

  If $\psi'(0)$ is contained in two distinct components E_i and E_{i+1}, $h(y'_1, \ldots, y'_d)$ is equal to

  $$\begin{aligned}(y'_1 y'_2, (y'_1)^{e_i}(y'_2)^{e_i+1}, (y'_1)^{e_i}(y'_2)^{e_i+1} y'_3, \ldots, (y'_1)^{e_i}(y'_2)^{e_i+1} y'_m,\\ (y'_1)^{e_i}(y'_2)^{e_i+1}\Phi_{m+1}, \ldots, (y'_1)^{e_i}(y'_2)^{e_i+1}\Phi_d) \, .\end{aligned}$$

  Here $\Phi = (\Phi_{m+1}, \ldots, \Phi_d)$ is a tuple of power series in the maximal ideal M of $k[[y'_1, \ldots, y'_d]]$, such that the determinant of $[\frac{\partial \Phi_i}{\partial y'_j}]_{i,j=m+1}^d$ is a unit at the origin. Furthermore, in the first case, each Φ_j is contained in the ideal $(y'_j) + M^2$, and in the second case, each Φ_j is contained in $(y'_1, y'_2, y'_j) + M^2$.
- *Either $\psi'(0) \notin X'$, and in this case we can choose $\Phi_j = y'_j$ for all j; or X' is locally defined by*

  $$y'_{m+1} = \ldots = y'_d = 0 \, .$$

Proof of the lemma. We prove the lemma when $\psi'(0) \in E_i \cap E_{i+1}$; the other case is easier. Choose local coordinates (y_i) on Y at $\psi(0)$ such that the order of $y_1(\psi)$ is minimal among $\{ord_t y_j(\psi)\}$. It follows from Hensel's lemma that we can reparametrize ψ, that is, compose ψ with an automorphism of $\operatorname{Spec} k[[t]]$, such that $y_1(\psi) = t^c$ for some positive integer c. After choosing new coordinates on our

ground space Y, we may assume that c does not divide the order of $y_j(\psi)$ if $j \neq 1$. Let c' be the minimum of $\{ord_t y_j(\psi) \,|\, j = 2, \ldots, d\}$. We may assume, changing coordinates if necessary, that $ord_t y_j(\psi) = c'$ iff $j = 2$. Since we never blow up points that belong to two distinct exceptional components, it is clear that e_i is equal to $\lfloor c/c' \rfloor$, and that we can find local coordinates (z'_i) at $\psi'(0)$ such that h is given by

$$h(z'_1, \ldots, z'_d) = (z'_1 z'_2, (z'_1)^{e_i}(z'_2)^{e_i+1}, (z'_1)^{e_i}(z'_2)^{e_i+1} z'_3, \ldots, (z'_1)^{e_i}(z'_2)^{e_i+1} z'_d)\,.$$

Since X', E_i and E_{i+1} intersect transversally at $\psi'(0)$, we can choose, after permutating the z'_j if necessary, new local coordinates (y'_i) at $\psi'(0)$, with $y'_j = z'_j$ for $j = 1, \ldots, m$, such that X' is defined by $y'_{m+1} = \ldots = y'_d = 0$. The part about the Φ_j is obvious. □

If $\psi'(0) \in X'$, it is clear that $\varphi = h(\varphi')$ is an optimal approximation for ψ in $\mathcal{L}(X)$, where $y'_i(\varphi') = y'_i(\psi')$ for $i = 1, \ldots, r-1$, and $y'_i(\varphi') = 0$ if $i \geq r$, and that the postulated formula for $d_X(\psi)$ holds. For suppose that η is an optimal approximation, and that η is a better approximation than φ. The fact that η lies at least as close to ψ as φ does (with respect to the valuation ord_t), guarantees that the lifting η' of η through h has its origin at $\psi'(0)$. It is clear that $y'_j(\eta') = 0$ for $j = m+1, \ldots, d$. Suppose that the minimum of $\{ord_t y'_j(\psi') \,|\, j = m+1, \ldots, d\}$ is realized for $j = d$.

If $\psi'(0)$ is contained in exactly one exceptional component, the fact that η is a better approximation than φ implies that

$$y'_1(\psi') \equiv y'_1(\eta') \, mod \, t^{e_i ord_t y'_1(\psi') + ord_t y'_d(\psi') + 1}\,,$$

and that

$$\Phi_d(y'_1(\psi'), \ldots, y'_d(\psi')) \equiv \Phi_d(y'_1(\eta'), \ldots, y'_d(\eta')) \, mod \, t^{ord_t y'_d(\psi') + 1}\,.$$

Since $\Phi_d \in (y'_d) + M^2$, and $\frac{\partial \Phi_d}{\partial y'_d} \neq 0$, and $y'_d(\eta') = 0$, we see that $ord_t(y'_j(\eta') - y'_j(\psi')) < ord_t y'_d(\psi')$ for some $j \in \{2, \ldots, m\}$, which makes φ a better approximation than η.

If $\psi'(0) \in E_i \cap E_{i+1}$, we can follow similar arguments: we see that

$$\begin{aligned}
y'_1(\psi')y'_2(\psi') &\equiv y'_1(\eta')y'_2(\eta') \, mod \, t^{e_i ord_t y'_1(\psi') + e_{i+1} ord_t y'_2(\psi') + ord_t y'_d(\psi') + 1}\,, \\
y'_1(\psi')^{e_i} y'_2(\psi')^{e_i+1} &\equiv y'_1(\eta')^{e_i} y'_2(\eta')^{e_i+1} \, mod \, t^{e_i ord_t y'_1(\psi') + e_{i+1} ord_t y'_2(\psi') + ord_t y'_d(\psi') + 1}\,, \\
\Phi_d(y'_1(\psi'), \ldots, y'_d(\psi')) &\equiv \Phi_d(y'_1(\eta'), \ldots, y'_d(\eta')) \, mod \, t^{ord_t y'_d(\psi') + 1}\,.
\end{aligned}$$

These observations again lead to the conclusion that $ord_t(y'_j(\eta') - y'_j(\psi')) < ord_t y'_d(\psi')$ for some $j \in \{2, \ldots, m\}$ (use the fact that for units u, v in $k[[t]]$, the congruence $u \equiv v \, mod \, t^a$ implies $u^{-1} \equiv v^{-1} \, mod \, t^a$, where $a \in \mathbb{N}$) .

Now assume that $\psi'(0) \notin X'$, and that $\psi'(0)$ is contained in exactly one exceptional component E_i. Let h_i be the blow-up of the point x, creating E_i, and decompose h as $h = \tilde{h} \circ h_i \circ h'$, where h and h' are compositions of blow-ups. It follows from the structure of the resolution h of X that there exists an arc φ' on X' such that $h_i \circ h' \circ \varphi'(0) = x$ and $y_1 \circ h(\varphi') = y_1(\psi)$, and such that E_i is the only exceptional component containing $h' \circ \varphi'(0)$. Let η be a non-constant optimal approximation of ψ in $\mathcal{L}(X)$. An argument similar to the one used above, shows that the lifting of η through $\tilde{h}$ has its origin at x. If the distance from η to ψ were strictly smaller than the distance from $\varphi = h(\varphi')$ to ψ, this would imply that the lifting of η through $\tilde{h} \circ h_i$ would lie in X', which contradicts our assumptions.

Hence, the formula for $d_X(\psi)$ holds in this case also. An analogous reasoning can be used when $\psi'(0)$ is contained in $E_i \cap E_{i+1} \setminus X$. $\square$

Using this formula, and the expression 1 in Section 3, it becomes very easy to compute the geometric Poincaré series for singularities with an embedded resolution satisfying the conditions of Theorem 1 .

Now suppose that X is a surface, and all conditions for Theorem 1 are satisfied, except that we allow E and X' to intersect non-transversally in a finite number of points x_i, each of which has to be a smooth point of E, at which the intersection multiplicity is two. We suppose that locally around x_i (with respect to the étale topology), the intersection of E and X' consists of two smooth prime divisors F_1 and F_2, meeting transversally at x_i (the obvious generalizations hold when the pair (X', E) is locally a product of a pair of the required form with a smooth space). The proof of Theorem 1 remains valid for each arc ψ whose lifting ψ' does not have its origin at one of the x_i.

We obtain transversal intersection by blowing up one of them, say F_1. Let $h' : \tilde{Y} \to Y'$ be the blow-up morphism, with exceptional divisor E_i', and let $\tilde{X}$ be the strict transform of X'. Suppose that the arc ψ lifts to an arc $\tilde{\psi}$, with origin on E, through the composition $h' \circ h$. Put ψ' equal to $h'(\tilde{\psi})$. Let $\tilde{F}$ be the inverse image of x_i under the projection $E_i' \cong F_1 \times \mathbb{P}_k^{d-2} \to F_1$. Blowing up F_2 yields an exceptional divisor E_i'', and a strict transform $\bar{X}$. Denote by $\bar{\psi}$ the lifting of $\tilde{\psi}$ through this blow-up morphism.

Let $\tilde{x}_i$ be the intersection of $\tilde{X}$ with $\tilde{F}$. By arguments similar to the ones in Lemma 2, we can find local coordinates $(\tilde{y}_1, \ldots, \tilde{y}_d)$ on $\tilde{Y}$ at $\tilde{x}_i$, local coordinates $(y_1', \ldots, y_d')$ on Y' at x_i, and local coordinates $(y_1, \ldots, y_d)$ on Y at $\psi(0)$, such that

- the morphism h is given in local coordinates by
$$h(y_1', \ldots, y_d') = (y_1', (y_1')^{e_i}\Phi_2, \ldots, (y_1')^{e_i}\Phi_d)\,,$$
where $\Phi_j \in (y_j) + M^2$ for each j, with M as in Lemma 2.
- the morphism h' is given in local coordinates by
$$h'(\tilde{y}_1, \ldots, \tilde{y}_d) = (\tilde{y}_1\tilde{y}_2, \tilde{y}_2, \tilde{y}_3, \tilde{y}_2\tilde{y}_4, \ldots, \tilde{y}_2\tilde{y}_d)\,,$$
- the strict transform of E_i under h' is defined by $\tilde{y}_1 = 0$,
- $\tilde{X}$ is defined by $\tilde{y}_3 = \tilde{y}_1$ and $\tilde{y}_4 = \ldots = \tilde{y}_d = 0$.
- The smooth germ X' is locally defined by the equations $y_1' - y_2'\,y_3' = 0$ and $y_4' = \ldots = y_d' = 0$.

Let H_i' be locally defined by $y_2' = 0$, and H_i'' by $y_3' = 0$. We use the same notation for their strict transforms under any blow-up.

First, we suppose $d = 3$. It follows from the expression for h, and an argument similar to the one used in the proof of Theorem 1, that we find an optimal approximation $\varphi = h(\varphi')$ by maximizing

$$\begin{aligned} d(\psi', \varphi') := \min\{&ord_t\,(y_1'(\varphi') - y_1'(\psi')),\, ord_t\,((y_1')^{e_i}(\varphi')y_2'(\varphi') - (y_1')^{e_i}(\psi')y_2'(\psi')), \\ &ord_t\,((y_1')^{e_i}(\varphi')y_3'(\varphi') - (y_1')^{e_i}(\psi')y_3'(\psi'))\}\,. \end{aligned}$$

After a suitable reparametrization, we may suppose that $y_1'(\psi') = t^N$. If the leading terms of $y_1'(\psi')$ and $y_1'(\varphi')$ differ, it is clear that we can replace φ' by another arc on X' whose image lies at least as close to ψ as φ does, whose leading y_1'-term agrees with that of ψ'. Hence, we might as well assume that they agreed all along.

Lemma 3. *It suffices to maximize*

$$\min\{ord_t\,(y_1'(\varphi')-y_1'(\psi')),\ e_i\,ord_t\,y_1'(\psi')+ord_t\,(y_2'(\varphi')-y_2'(\psi')),\\ e_i\,ord_t\,y_1'(\psi')+ord_t\,(y_3'(\varphi')-y_3'(\psi'))\}\,.$$

This can always be achieved without modifying y_1', *i.e. we can put* $y_1'(\varphi')=y_1'(\psi')$.

Proof of the lemma. Given an approximation φ' in $\mathcal{L}(X')_{x_i}$, with $y_1'(\varphi')\neq y_1'(\psi')=t^N$, we construct an arc η' in $\mathcal{L}(X')_{x_i}$, such that $y_1'(\eta')=t^N$, and such that $d(\eta',\psi')\geq d(\varphi',\psi')$.

We already noticed that we may suppose $y_1'(\varphi')=t^N+t^{N+1}\phi'$. Now define an arc η' by $y_1'(\eta')=t^N$, $y_2'(\eta')=y_2'(\varphi')(1+t\phi')^{-1}$, and $y_3'(\eta')=y_3'(\varphi')$. Since $d(\psi',\varphi')$ is at most $N+1+ord_t\,\phi'$, it is sufficient to show that $d(\eta',\varphi')$ is at least $N+1+ord_t\,\phi'$. However, this is clear from the definition. □

If $\tilde\psi(0)\notin\tilde X$, the order of $y_2'(\psi')$ is larger than, or equal to the order of $y_1'(\psi')$. If $ord_t\,y_3'(\psi')$ is smaller than $\lceil ord_t\,y_1'(\psi')/2\rceil$, we define $y_2'(\varphi')$ to be $y_1'(\psi')(y_3'(\psi'))^{-1}$. In the other case, we take for $y_2'(\varphi')$ and $y_3'(\varphi')$ two power series of order $\lfloor ord_t\,y_1'(\psi')/2\rfloor$, resp. $\lceil ord_t\,y_1'(\psi')\rceil/2$, whose product equals $y_1'(\psi')$. So

$$d_X(\psi)=e_ic(\tilde\psi,E_i')+\max\{\lfloor c(\tilde\psi,E_i')/2\rfloor,\ c(\tilde\psi,E_i')-c(\tilde\psi,H_i'')\}\,.$$

Suppose that $\tilde\psi(0)$ is contained in $\tilde X$. If the leading terms of $y_1'(\psi')$ and $y_2'(\psi')y_3'(\psi')$ differ, analogous arguments yield

$$d_X(\psi)=e_ic(\psi',E_i)+\max\{\min\{\lfloor c(\psi',E_i)/2\rfloor,c(\psi',H_i'),c(\psi',H_i'')\},\\ \min\{c(\psi',H_i'),c(\psi',E_i)-c(\psi',H_i'')\},\ \min\{c(\psi',H_i''),\ c(\psi',E_i)-c(\psi',H_i')\}\}\,.$$

When the leading terms of $y_1'(\psi')$ and $y_2'(\psi')y_3'(\psi')$ coincide,

$$\begin{aligned} d_X(\psi) &= e_i ord_t\,y_1'+d_{X'}(\psi')-\min\{c(\psi',H_i'),\ c(\psi',H_i'')\}\\ &= e_ic(\tilde\psi,E_i)+(e_i+1)c(\tilde\psi,E_i')+d_{\tilde X}(\tilde\psi)-c(\tilde\psi,\tilde F)\,.\end{aligned}$$

Blowing up F_2, we see that the following formula holds in general:

$$\begin{aligned} d_X(\psi) &= d_{\bar X}(\bar\psi)+e_ic(\bar\psi,E_i)+e_ic(\bar\psi,E_i')+e_ic(\bar\psi,E_i'')+\\ &\quad+\max\{\min\{\lfloor c(\bar\psi,E_i)/2+c(\bar\psi,E_i')/2+c(\bar\psi,E_i'')/2\rfloor,\\ &\qquad\qquad c(\bar\psi,E_i')+c(\bar\psi,H_i'),c(\bar\psi,E_i'')+c(\bar\psi,H_i'')\},\\ &\quad\min\{c(\bar\psi,E_i')+c(\bar\psi,H_i'),\ c(\bar\psi,E_i)+c(\bar\psi,E_i')-c(\bar\psi,H_i'')\},\\ &\quad\min\{c(\bar\psi,E_i'')+c(\bar\psi,H_i''),\ c(\bar\psi,E_i)+c(\bar\psi,E_i'')-c(\bar\psi,H_i')\}\}\,. \end{aligned} \tag{2}$$

Now suppose $d>3$. Let $\bar Z$ be the strict transform of the étale germ Z' at x_i, locally defined by $y_1'-y_2'y_3'=0$, and let W' be the germ at x_i defined by $y_4'=\ldots=y_d'=0$. Provided that we replace $d_{\bar X}(\bar\psi)$ by $d_{\bar Z}(\bar\psi)$, our formula remains valid, unless $e_ic(E_i)+ord_t\,y_j'$ is smaller than the above expression (2), for some j in $\{4,\ldots,m\}$. If this is the case, $d_X(\psi)$ will be equal to $e_ic(\psi',E_i)+d_{W'}(\psi')$.

We've proven

Theorem 2. *The valuation $d_X(\psi)$ is the minimum of*

$$\begin{aligned}&d_{\bar Z}(\bar\psi)+e_ic(\bar\psi,E_i)+e_ic(\bar\psi,E_i')+e_ic(\bar\psi,E_i'')+\\&\quad+\max\{\,\min\{\lfloor c(\bar\psi,E_i)/2+c(\bar\psi,E_i')/2+c(\bar\psi,E_i'')/2\rfloor,\\&\qquad\qquad c(\bar\psi,E_i')+c(\bar\psi,H_i'),c(\bar\psi,E_i'')+c(\bar\psi,H_i'')\},\\&\quad\min\{c(\bar\psi,E_i')+c(\bar\psi,H_i'),\ c(\bar\psi,E_i)+c(\bar\psi,E_i')-c(\bar\psi,H_i'')\},\\&\quad\min\{c(\bar\psi,E_i'')+c(\bar\psi,H_i''),\ c(\bar\psi,E_i)+c(\bar\psi,E_i'')-c(\bar\psi,H_i')\}\,\}\,.\end{aligned}$$

and $d_{W'}(\psi')+e_ic(\psi',E_i)$.

Let us compute the motivic integral

$$\begin{aligned}D^{(e_i)}(X,s)&=\int_{\mathcal{L}(Y')_{x_i}}\mathbb{L}^{-d_X\circ h(\psi')s-ord_t\,Jac_h}d\mu(\psi')\\&=\int_{\mathcal{L}(\tilde Y)_{h'^{-1}(x_i)}}\mathbb{L}^{-d_X\circ h\circ h'(\tilde\psi)s-ord_t\,Jac_{h\circ h'}}d\mu(\tilde\psi)\,.\end{aligned}$$

You can visualize the situation by taking E_i' and E_i'' to be two walls making a rectangular corner, and by imagining $\bar X$, E_i and H_i'' to be horizontal shelves. The fiber over $\tilde x_i$ is equal to $(E_i'\cap E_i'')$. To simplify notation, we denote $e_i(d-1)+1$ by ν_1, and $(e_i+1)(d-1)$ by ν_2. The order of the Jacobian of the resolution morphism on E_i is ν_1, and on E_i', as well as on E_i'', it equals ν_2.

First, suppose $d=3$. The easiest way to compute $D^{(e_i)}$ would be to take a shortcut and use the formula in [17] for the geometric Poincaré series for toric surfaces. The minimal resolution of the toric hypersurface defined by $xz-y^{2e_i+1}=0$ has a point of non-transversal intersection at depth e_i. Computing the contribution of the remainder of the exceptional locus to P_{geom}, as is done in Section 7, allows us to derive the contribution of this point, hence the motivic integral $D^{(e_i)}$. However, we prefer a direct computation, in order to obtain a shorter and more elementary method to compute the geometric Poincaré series of a toric surface singularity.

First, we take care of arcs ψ for which $\tilde\psi\notin\tilde X$. Their contribution equals $\mathbb{L}^{-2}(\mathbb{L}-1)$ times

$$\frac{(\mathbb{L}-1)\mathbb{L}^{-(3e_i+2)s-3\nu_2-1}}{(1-\mathbb{L}^{-(e_i+1)s-\nu_2})(1-\mathbb{L}^{-(2e_i+1)s-2\nu_2-1})}+\mathbb{L}^{-e_is-\nu_2}\frac{1+\mathbb{L}^{-(e_i+1)s-\nu_2}}{1-\mathbb{L}^{-(2e_i+1)s-2\nu_2-1}}\,.$$

The integral over $(E_i'\cap E_i'')\setminus(E_i\cup H_i'')$ is equal to $\mathbb{L}^{-3}(\mathbb{L}-1)^2$ times

$$(\mathbb{L}-2+\frac{(\mathbb{L}-1)\mathbb{L}^{-s-1}}{1-\mathbb{L}^{-s-1}})\frac{\mathbb{L}^{-(2e_i+1)s-2\nu_2}}{1-\mathbb{L}^{-(2e_i+1)s-2\nu_2}}\frac{1+\mathbb{L}^{-(e_i+1)s-\nu_2}}{1-\mathbb{L}^{-(e_i+1)s-\nu_2}}\,.$$

We blow up $E_i'\cap E_i''$ to compute the contribution of $H_i''\cap E_i'\cap E_i''$. Let G be the exceptional divisor. We denote the strict transforms of $\bar E_i$, E_i' and E_i'' again by the same symbols. The question is how

$$\begin{aligned}\max\{&\min\{\lfloor c(\bar\psi,E_i')/2+c(\bar\psi,E_i'')/2\rfloor,c(\bar\psi,E_i'),c(\bar\psi,E_i'')+c(\bar\psi,H_i'')\},\\&\qquad c(\bar\psi,E_i'),\ c(\bar\psi,E_i')-c(\bar\psi,H_i'')\,\}\end{aligned}$$

behaves on G.

Straightforward computation shows that the term associated to $H_i''\cap E_i'\cap E_i''$ is equal to

$$\mathbb{L}^{-3}(\mathbb{L}-1)^2\frac{\mathbb{L}^{-(2e_i+1)s-2\nu_2}}{1-\mathbb{L}^{-(2e_i+1)s-2\nu_2}}$$

times

$$1+\frac{\mathbb{L}^{-(e_i+1)s-\nu_2}}{1-\mathbb{L}^{-(e_i+1)s-\nu_2}}+\frac{(\mathbb{L}-1)\mathbb{L}^{-(3e_i+2)s-3\nu_2-1}}{(1-\mathbb{L}^{-(e_i+1)s-\nu_2})(1-\mathbb{L}^{-(2e_i+1)s-2\nu_2-1})}$$
$$+\mathbb{L}^{-e_is-\nu_2}\frac{1+\mathbb{L}^{-(e_i+1)s-\nu_2}}{1-\mathbb{L}^{-(2e_i+1)s-2\nu_2-1}}\,.$$

Finally, we compute the contribution of $E_i\cap E_i'\cap E_i''$. Our expression for $d_X(\psi)$ reduces to

$$e_ic(\bar{\psi},E_i)+e_ic(\bar{\psi},E_i')+e_ic(\bar{\psi},E_i'')+\max\{c(\bar{\psi},E_i'),c(\bar{\psi},E_i'')\}\,.$$

Hence, we get

$$\mathbb{L}^{-3}(\mathbb{L}-1)^3\frac{\mathbb{L}^{-(3e_i+1)s-2\nu_2-\nu_1}}{(1-\mathbb{L}^{-(2e_i+1)s-2\nu_2})(1-\mathbb{L}^{-e_is-\nu_1})}\frac{1+\mathbb{L}^{-(e_i+1)s-\nu_2}}{1-\mathbb{L}^{-(e_i+1)s-\nu_2}}\,.$$

Bringing all these terms together, we see that

$$D^{(e_i)}(X,s)=\mathbb{L}^{-3}(\mathbb{L}-1)\frac{1+\mathbb{L}^{-(e_i+1)s-\nu_2}}{1-\mathbb{L}^{-(2e_i+1)s-2\nu_2}}\{\frac{(\mathbb{L}-1)^2\mathbb{L}^{-(2e_i+1)s-2\nu_2}}{(1-\mathbb{L}^{-s-1})(1-\mathbb{L}^{-e_is-\nu_1})}+\mathbb{L}^{-e_is-\nu_1}\}\,.$$

The contribution of x_i to P_{geom}, i.e.

$$-\frac{\mathbb{L}^3D^{(e_i)}(\mathbb{L}^3T)}{1-\mathbb{L}^3T}\,,$$

is equal to

$$-\frac{(\mathbb{L}-1)(1+\mathbb{L}^{e_i+1}T^{e_i+1})}{(1-\mathbb{L}^3T)(1-\mathbb{L}^{2e_i-1}T^{2e_i+1})}\{\frac{(\mathbb{L}-1)^2\mathbb{L}^{2e_i-1}T^{2e_i+1}}{(1-\mathbb{L}^2T)(1-\mathbb{L}^{e_i-1}T^{e_i})}+\mathbb{L}^{e_i-1}T^{e_i}\}\,.$$

Now suppose $d>3$. Consider the singular surface U in $\mathbb{A}^3_k$, defined by $yz-x^{2e_i+1}=0$. We consider $\mathbb{A}^d_k$ as the product of $\mathbb{A}^3_k$ and $\mathbb{A}^{d-3}_k$, and we identify $\mathbb{A}^3_k$ with $\mathbb{A}^3_k\times 0$. The minimal toric resolution of U is induced by a succession $g':A'\to\mathbb{A}^3_k$ of blow-ups of points in $\mathbb{A}^3_k$, see Section 7. Blowing up the same points in $\mathbb{A}^d_k$ yields a morphism $g:A\to\mathbb{A}^d_k$. The geometric Poincaré series $\tilde{P}_{U,geom}(T)$ of U at the origin O is defined intrinsically, hence does not depend on the embedding in ambient space. It is computed in Section 7, only using our formula for $d=3$.

Applying the change of variables formula to the proper birational morphism g, we write $D_O(U,s)$ as $D^{(e_i)}(U,s)+R(s)$, where $D^{(e_i)}(U,s)$ is, of course, the integral over the arcs in the resolution space with origin at the unique point where the strict transform of U and the exceptional locus intersect non-transversally. The term R is the motivic integral over the remainder of the fiber of g over the origin.

It follows immediately from our formula that $D^{(e_i)}(U,s)=D^{(e_i)}(X,s)$. The formula in Lemma 1 yields

$$P_{U,geom}(T)=\frac{1-\mathbb{L}^d(D^{(e_i)}(\mathbb{L}^dT)+R(\mathbb{L}^dT))}{1-\mathbb{L}^dT}\,.$$

The term $\mathbb{L}^dR(\mathbb{L}^dT)$ can be easily computed, as is done for $d=3$ in Section 7. Writing e_i as $2m_i+1$, and d as $t+2$, we see that

$$\mathbb{L}^dD^{(e_i)}(\mathbb{L}^dT)$$

is equal to

$$
\begin{aligned}
-(1-\mathbb{L}^{t+2}T)&\frac{(\mathbb{L}-1)\sum_{j=1}^{m_i-1}\mathbb{L}^jT^j}{(1-\mathbb{L}^2T)(1-\mathbb{L}^{m_j-1}T^{m_j})}+(1-\mathbb{L}^{t+2}T)\frac{(\mathbb{L}-1)\mathbb{L}T}{(1-T)(1-\mathbb{L}^2T)}\\
&+\frac{\mathbb{L}^2(\mathbb{L}^t-1)(\mathbb{L}-1)T^2}{(1-T)(1-\mathbb{L}^2T)}-\frac{(\mathbb{L}-1)^2([\mathbb{P}^t]-2)\sum_{i=1}^{m_j-1}\mathbb{L}^iT^{i+1}}{(1-T)(1-\mathbb{L}^{m_j-1}T^{m_j})}\\
&-2\frac{(\mathbb{L}-1)^2(\mathbb{L}^t-1)\mathbb{L}^2T\sum_{i=1}^{m_j-1}\mathbb{L}^iT^{i+1}}{(1-T)(1-\mathbb{L}^{m_j-1}T^{m_j})(1-\mathbb{L}^2T)}\,.
\end{aligned}
$$

6. Quasirational singularities

The theorems in the previous section allow us to give a partial answer to the question raised by Lejeune-Jalabert and Reguera-Lopez in [17].

Lemma 4. *Let C be an irreducible curve over k. If the class $[C]$ of C in the Grothendieck ring belongs to $\mathbb{Z}[\mathbb{L}]$, then C is rational.*

Proof. It follows from Riemann-Roch that a smooth projective curve of genus 0 is rational [15]. The Poincaré polynomial $P(u)$ is an additive invariant, i.e. it is well-defined on $K_0(Var_k)$, since it is obtained from the Hodge polynomial by identifying the two variables. For a complete smooth curve X, the polynomial $P[X]$ is equal to $u^2-2g(X)u+1$, where $g(X)$ is the genus of X.

A curve C is rational if and only if its projective smooth birational model $\bar{C}$ is. Furthermore, $[C]\in\mathbb{Z}[\mathbb{L}]$ implies $[\bar{C}]\in\mathbb{Z}[\mathbb{L}]$. It follows from the identities $P(\mathbb{L})=u^2$ and $P(1)=1$ that the linear term of $P[\bar{C}]$ is zero, hence $\bar{C}$ is rational, and so is C. □

Theorem 3. *Let x be an isolated singularity of a surface $X\subset Y$, with Y smooth, and assume that there exists an embedded resolution of the germ of X at x, satisfying the conditions of Theorem 1 or 2. The local Poincaré series of X at x can be written as a rational function with numerator and denominator in $\mathbb{Z}[\mathbb{L}][T]$, if and only if x is a quasirational singularity.*

Let us recall that a surface singularity (X,x) is quasirational if "only rational curves can come out of the singularity, no matter how we blow it up birationally" [1]. Rational surface singularities are quasirational.

Proof. One implication is straightforward: suppose that x is a quasirational singularity. It suffices to write P_{geom} in terms of the motivic integral $D(s)$, and to observe that the quasirationality of x implies that all Grothendieck brackets of the strata of the exceptional locus, emerging in the expression for P_{geom}, belong to $\mathbb{Z}[\mathbb{L}]$.

Let e_i be the minimal depth of a global exceptional component E_i in the resolution of (X,x), at which a non-rational exceptional component appears. We will prove that the coefficient A_i of T^{e_i} in P_{geom} is not contained in $\mathbb{Z}[\mathbb{L}]$.

It is clear from our formulae that A_i is equal to the sum of a term in $\mathbb{Z}[\mathbb{L}]$ with $(\mathbb{L}-1)\sum\mathbb{L}^{\mu_i}[C_i]$, where the μ_i are positive integers, and we take the sum over all non-rational exceptional components C_i on X' which are contained in an exceptional divisor E_i of depth e_i. Since the coefficient of the linear term of $P[C_i]$ is strictly greater than zero, this sum can not be contained in $\mathbb{Z}[\mathbb{L}]$, for its Poincaré polynomial will contain a term of odd degree. □

7. Toric surfaces

Theorem 1 provides an elementary method to compute the geometric Poincaré series of a toric surface singularity, which is substantially shorter than the techniques used in [17].

Throughout this section, X is a singular affine toric surface, defined by a cone σ generated by $(1,0)$ and (p,q), where $0<p<q$ and p,q are relatively prime. Let $(b_1,\ldots,b_s)$ be the entries occurring in the Hirzebruch-Jung continued fraction associated to $q/(q-p)$, and $(c_1,\ldots,c_t)$ the components of the continued fraction of q/p [12][19]. The relation between the b_i and the c_j is explained in [18]. Let furthermore Θ be the union of compact faces of the convex hull of $\sigma\cap N\setminus 0$, and $\check{\Theta}$ be the union of compact faces of the convex hull of $\check{\sigma}\cap M\setminus 0$.

The minimal resolution of X is a toric modification induced by a subdivision of σ into simple cones. The vectors occurring in this subdivision can be listed as follows:

$$v_0=(1,0),\, v_1=(1,1),\ldots,\, v_{j+1}=b_jv_j-v_{j-1},\ldots,\, v_{s+1}=b_sv_s-v_{s-1}=(p,q)\,.$$

The exceptional divisors $D_j\cong\mathbb{P}^1$ of this resolution correspond to the newly introduced vectors v_j, $j=1,\ldots,s$, and D_j is known to have self-intersection number $-b_j$.

The c_j have a geometric significance of their own: subdividing $\check{\sigma}$ into simple cones, i.e. taking the minimal set of generators for the semi-group $\check{\sigma}\cap M$, yields an embedding of X into affine $(t+2)$-space; the ideal of X is generated by $x_{i-1}x_{i+1}-x_i^{c_i}$, $i=1,\ldots,t$.

We will factor the canonical toric resolution into a sequence of blow-ups of zero-dimensional orbits, which can be immediately extended to an embedded resolution for X using the embedding in affine space mentioned above. Blowing up the unique zero-dimensional orbit O of V corresponds, by [16], to the toric modification corresponding to the subdivision Σ of σ introducing all primitive vectors normal to the edges of $\check{\Theta}$. This comes down to inserting v_1, v_{s-1}, and all v_i determining vertices of Θ, i.e. the v_i for which $b_i\neq 2$. Let a be the number of vectors introduced in Σ, i.e. the number of elements in $\{b_2,\ldots,b_{s-1}\}$ differing from 2 augmented by two, and let $b=a-r-1$ be the number of pairs of adjacent vectors in Σ, that is, pairs of vectors in Σ with multiplicity 1. The number b is equal to the number of c_j equal to 3, while r is equal to the cardinality of $\{c_j>3\}$.

The singularities left after blowing up O are all rational singularities of type A_c. In fact, they are recovered from the b_i by omitting b_1 and b_s, and isolating all sequences of 2's in the remaining b_i. Let c be the number of 2's in such a sequence. This number c can be recovered from the c_j: it is equal to c_j-3, with j chosen such that the vertex of $\check{\Theta}$ corresponding to x_j lies on the two edges whose normal directions determine the cone in our fan Σ corresponding to this sequence of 2's. Moreover, each of the c_j which is bigger than 3 will induce a singularity in this way. The singularity will be resolved after blowing up the zero-dimensional orbit corresponding to the associated singular cone (thus inserting 2 vectors, or 1 if $c=1$) and repeating this procedure $\lfloor c/2\rfloor$ times. If c is even, we get a chain of exceptional divisors intersecting $\tilde{X}$ transversally; if d_k is odd, we get an intersection point of multiplicity 2 in the last stage of the resolution process.

This factorization allows us to embed our resolution in ambient affine space, simply by blowing up the corresponding points in this space. Let $h:\tilde{Y}\to Y=\mathbb{A}_k^{t+2}$

be the proper birational morphism obtained in this way, and let $\tilde{X}$ be the strict transform of X (thus $\tilde{X}$ is the canonical resolution surface). The points of $\tilde{X}$ where there's no transversal intersection with the exceptional locus of h correspond to adjacent vectors in the simple subdivision of σ which are introduced in one and the same blow-up.

In [18], we used this embedded resolution to compute the motivic Igusa Poincaré series of X. Our computation of P_{geom} will be very similar.

Define E_{-1} to be the strict transform of X under h. Let E_0 be the strict transform of the exceptional divisor that is created in the first blow-up, and let $E_{i,j}$ be the strict transform of the exceptional divisor induced by the j-th blow-up of the singularity corresponding to the i-th sequence of 2's in $b_2, \dots, b_{s-1}$.

Let c_i' be the i-th component of $(c_1, \dots, c_t)$ which is strictly larger than 3, and put d_i equal to $c_i' - 3$. We let I denote the index set

$$\{-1, 0\} \cup \{(i,j) \,|\, i \in \{1, \dots, r\},\, j \in \{1, \dots, \lceil d_i/2 \rceil\}\}\,.$$

We stratify $\tilde{X}$ in the usual way: for each subset J of I, we define E_J to be $\cap_{\alpha \in J} E_\alpha$, while E_J^o denotes $E_J \setminus \cup_{\alpha \notin J} E_\alpha$.

We attach to each E_α a pair of numerical data (N_α, ν_α) as follows:

$$(N_{-1}, \nu_{-1}) = (1, t),\, (N_0, \nu_0) = (1, t+2), (N_{(i,j)}, \nu_{(i,j)}) = (j+1, (j+1)(t+1)+1)\,.$$

Then

$$\begin{aligned} D(s) \quad = \quad & \mathbb{L}^{-(t+2)} \sum_{J \subset I, J \not\subseteq \{-1\}} [E_J^o] \prod_{\alpha \in J} \frac{(\mathbb{L}^{codim\, E_\alpha} - 1)\mathbb{L}^{-N_\alpha s - \nu_\alpha}}{1 - \mathbb{L}^{-N_\alpha s - \nu_\alpha}} \\ & + b\, \mathbb{L}^{-(t+2)} \{ D^{(1)}(s) - \frac{(\mathbb{L}^t - 1)(\mathbb{L} - 1)\mathbb{L}^{-3s-2t-2}}{(1 - \mathbb{L}^{-s-t})(1 - \mathbb{L}^{-2s-t-2})} \} \\ & + \sum_{d_i \ even} \{ D^{(d_i/2+1)}(s) - \frac{(\mathbb{L}^t - 1)(\mathbb{L} - 1)\mathbb{L}^{-(d_i/2+2)s - (d_i/2+2)(t+1)}}{(1 - \mathbb{L}^{-s-t})(1 - \mathbb{L}^{-(d_i/2+1)s - (d_i/2+1)(t+1) - 1})} \} \end{aligned}$$

The last terms in the expression for $D(s)$ correct for non-transversal intersection. We refer to [18] for a more detailed description of the terms.

We immediately recover a result from [17], stating that P_{geom} is trivial when all c_j are equal to 2, i.e. that P_{geom} equals the local geometric series of a smooth point. More generally, we can state the following corollary of Theorem 1:

Corollary 1. *Let $X \subset Y$ be varieties over k, where Y is smooth, and X has dimension m. Let x be an isolated singularity of X, and let $h : Y' \to Y$ be a composition of blow-ups of points, satisfying the conditions of Theorem 1, with exceptional divisor E, which is a linear chain of components E_i. Let X' be the strict transform of X. Assume that, for each i, the class of $E_i \cap X'$ in $K_0(Var_k)$ is equal to $[\mathbb{P}_k^{m-1}]$, and $[E_i \cap E_{i+1} \cap X] = [\mathbb{P}_k^{m-2}]$. Then P_{geom} is trivial.*

To put it intuitively: if the embedded resolution looks like X was smooth at x all along, P_{geom} cannot distinguish x from a smooth point. The general idea is that similar embedded resolution graphs yield similar geometrical Poincaré series.

Let us reduce the formula for $P_{geom}(T)$ to the expression given in [17]. First, we treat the case $t = 1$, that is, we compute the geometric Poincaré series associated to the toric surface singularity defined by $xz = y^{c_j}$. We already know what happens for $c_j = 2$, so we may suppose $c_j > 2$. Let $D'(s)$ be the sum of terms

of $D(s)$ corresponding to index sets not containing any of the couples (i,j). The contribution
$$\frac{1-\mathbb{L}^d D'(\mathbb{L}^d T)}{1-\mathbb{L}^d T}$$
is equal to
$$\frac{1}{1-\mathbb{L}^2T}+\frac{(\mathbb{L}-1)\mathbb{L}T}{(1-T)(1-\mathbb{L}^2T)}+\frac{\mathbb{L}^2(\mathbb{L}-1)^2T^2}{(1-T)(1-\mathbb{L}^2T)(1-\mathbb{L}^3T)}\,.$$
If $c_j=3$, we have to add the contribution
$$-\frac{(\mathbb{L}-1)(1+\mathbb{L}^2T^2)}{(1-\mathbb{L}^3T)(1-\mathbb{L}T^3)}\{\frac{(\mathbb{L}-1)^2\mathbb{L}T^3}{(1-\mathbb{L}^2T)(1-T)}+T\}\,,$$
and we obtain
$$P_{geom}(T)=\frac{1}{1-\mathbb{L}^2T}+(\mathbb{L}-1)\frac{(\mathbb{L}-1)T+\mathbb{L}T^2}{(1-\mathbb{L}^2T)(1-\mathbb{L}T^3)}\,.$$

If $c_j>3$, write c_j as $2m_j+n_j$, where m_j is a positive integer, and n_j is either zero or one.

First, we consider the strata $[E^o_{(i,j)}]=(\mathbb{L}-1)^2$ and $[E^o_{\{0,(1,j)\}}]=[E^o_{\{(i,j),(i+1,j)\}}]=\mathbb{L}-1$. The corresponding terms in the expression for P_{geom} amount to
$$-\frac{(\mathbb{L}-1)^3}{1-\mathbb{L}^3T}\{\frac{\mathbb{L}T^3}{(1-T)(1-\mathbb{L}T^2)}+\sum_{i=2}^{m_j-1}\frac{\mathbb{L}^{i-1}T^i}{(1-\mathbb{L}^{i-1}T^i)(1-\mathbb{L}^iT^{i+1})}\}-\frac{(*)^{(j)}}{1-\mathbb{L}^3T}$$
where the term $(*)_j$ depends on the value of n_j. The part between braces is easily seen to be equal to
$$\frac{\mathbb{L}^{m_j-1}T^{m_j+1}+\sum_{i=1}^{m_j-2}\mathbb{L}^iT^{i+1}}{(1-T)(1-\mathbb{L}^{m_j-1}T^{m_j})}\,.$$

If $n_j=0$, the term $(*)^{(j)}$ equals
$$(\mathbb{L}-1)(\mathbb{L}^2-\mathbb{L}+1)\frac{\mathbb{L}^{m_j-1}T^{m_j}}{1-\mathbb{L}^{m_j-1}T^{m_j}}\,,$$
while, if $n_j=1$, it is equal to
$$(\mathbb{L}-1)^3\frac{\mathbb{L}^{m_j-1}T^{m_j}}{1-\mathbb{L}^{m_j-1}T^{m_j}}\,.$$
Hence, we get
$$-\frac{(\mathbb{L}-1)^3\sum_{i=1}^{m_j-1}\mathbb{L}^iT^{i+1}}{(1-\mathbb{L}^3T)(1-T)(1-\mathbb{L}^{m_j-1}T^{m_j})}+(n_j-1)\frac{\mathbb{L}(\mathbb{L}-1)\mathbb{L}^{m_j-1}T^{m_j}}{(1-\mathbb{L}^3T)(1-\mathbb{L}^{m_j-1}T^{m_j})}\,.$$

Now, we look at the remaining strata, except for the points where we get non-transversal intersection. This yields, in exactly the same way,
$$-2\frac{(\mathbb{L}-1)^3\mathbb{L}^2T}{(1-\mathbb{L}^2T)(1-\mathbb{L}^3T)}\frac{\sum_{i=1}^{m_j-1}\mathbb{L}^iT^{i+1}}{(1-T)(1-\mathbb{L}^{m_j-1}T^{m_j})}$$
$$-(n_j-1)\frac{(\mathbb{L}-1)^3\mathbb{L}^{m_j+1}T^{m_j+1}}{(1-\mathbb{L}^{m_j-1}T^{m_j})(1-\mathbb{L}^2T)(1-\mathbb{L}^3T)}\,.$$
If c_j is even, we see that $P_{geom}(T)$ equals
$$\frac{1}{1-\mathbb{L}^2T}+(\mathbb{L}-1)\frac{\sum_{i=1}^{m_j-1}\mathbb{L}^iT^i}{(1-\mathbb{L}^2T)(1-\mathbb{L}^{m_j-1}T^{m_j})}\,.$$

If c_j is odd, we have to include the contribution of the point where there's no transversal intersection with the exceptional locus. This yields

$$P_{geom}(T) = \frac{1}{1-\mathbb{L}^2T} + (\mathbb{L}-1)\frac{(1+\mathbb{L}^{m_j-1}T^{m_j})\sum_{i=1}^{m_j}\mathbb{L}^iT^i - \mathbb{L}^{m_j-1}T^{m_j}}{(1-\mathbb{L}^2T)(1-\mathbb{L}^{2m_j-1}T^{2m_j+1})} \, .$$

To conclude, let us consider the case $d > 3$. It follows from Theorems 1 and 2, our computations for $d = 3$, and the fact that the geometric Poincaré series is defined intrinsically, that the contribution to P_{geom} of all strata, except for $E^o_{\{0\}}$ and $E^o_{\{-1,0\}}$, is equal to

$$\begin{aligned} &-(a-1)\frac{(\mathbb{L}-1)\mathbb{L}T}{(1-T)(1-\mathbb{L}^2T)} - (a-1)\frac{\mathbb{L}^2(\mathbb{L}^t-1)(\mathbb{L}-1)T^2}{(1-T)(1-\mathbb{L}^2T)(1-\mathbb{L}^{t+2}T)} \\ &\quad + \sum_{c_j>2 \ even}(\mathbb{L}-1)\frac{\sum_{i=1}^{m_j-1}\mathbb{L}^iT^i}{(1-\mathbb{L}^2T)(1-\mathbb{L}^{m_j-1}T^{m_j})} \\ &\quad + \sum_{c_j>2 \ odd}(\mathbb{L}-1)\frac{(1+\mathbb{L}^{m_j-1}T^{m_j})\sum_{i=1}^{m_j}\mathbb{L}^iT^i - \mathbb{L}^{m_j-1}T^{m_j}}{(1-\mathbb{L}^2T)(1-\mathbb{L}^{2m_j-1}T^{2m_j+1})} \, . \end{aligned}$$

This observation allows us to conclude that

$$\begin{aligned} P_{geom}(T) \quad = \quad & \frac{1}{1-\mathbb{L}^2T} + \sum_{c_j>2 \ even}(\mathbb{L}-1)\frac{\sum_{i=1}^{m_j-1}\mathbb{L}^iT^i}{(1-\mathbb{L}^2T)(1-\mathbb{L}^{m_j-1}T^{m_j})} \\ & + \sum_{c_j>2 \ odd}(\mathbb{L}-1)\frac{(1+\mathbb{L}^{m_j-1}T^{m_j})\sum_{i=1}^{m_j}\mathbb{L}^iT^i - \mathbb{L}^{m_j-1}T^{m_j}}{(1-\mathbb{L}^2T)(1-\mathbb{L}^{2m_j-1}T^{2m_j+1})} \, . \end{aligned}$$

It follows from results in [5] that this formula holds not only over $\hat{\mathcal{M}}_k$, but already over $\mathcal{M}_k$. Of course, this formula also holds for surface singularities with "the same embedded resolution graph" as a toric surface singularity.

8. The arithmetic series

In this section, k is an arbitrary field of characteristic zero, not necessarily algebraically closed, and we denote by k^{alg} an algebraic closure. Let X be a variety over k, and let x be a point of $X(k)$. As we've seen, the local geometric Poincaré series counts the jets in $\mathcal{L}_n(X)_x$, which can be lifted to an arc in $\mathcal{L}(X)_x$. However, working scheme-theoretically, we allow extensions of our base field k in this lifting process (which is necessary to ensure that $[\pi_n\mathcal{L}(X)_x]$ is well-defined). Hence, $P_{geom}(T)$ is insensitive to issues of rationality. These are taken into account by the arithmetic series $P_{arith}(T)$.

Bittner [2] gave a short proof of the existence of a ring morphism χ_{mot} from the Grothendieck ring $K_0(Var_k)$ of varieties over k to the Grothendieck ring $K_0(CH_k)$ of Chow motives over k, sending the class of a smooth projective variety to the class of its associated Chow motive, and sending $\mathbb{L}$ to the class of the Tate motive $\mathbb{L}_{mot}$. The existence of this map was originally proven in [14]. In [8], Denef and Loeser constructed a morphism

$$\chi_c : K_0(PFF_k) \to K_0^{mot}(Var_k) \otimes \mathbb{Q} \, ,$$

where $K_0^{mot}(Var_k)$ is the image of $K_0(Var_k)$ under the morphism χ_{mot}, and $K_0(PFF_k)$ is the Grothendieck group of pseudo-finite fields containing k. Elements of $K_0(PFF_k)$ are equivalence classes of ring formulas over k.

For this construction, it is important to understand the structure of $K_0(PFF_k)$. The theory of quantifier elimination for pseudo-finite fields [10][11], states that quantifiers can be eliminated if one adds some relations to the language, which have a geometric interpretation in terms of Galois covers. This interpretation yields a construction for χ_c. It is important for our purposes that, if our original ring formula φ did not contain any quantifiers in the first place, χ_c maps $[\varphi]$ to the class of the constructible set defined by φ in $K_0^{mot}(Var_k)$. We refer to [18] for a short introduction to the arithmetic series, and to [7][8] for the original work on arithmetic integration.

Let us define the local arithmetic Poincaré series P_{arith} of X at x. Since we're working locally, we may assume that X is a subvariety of some affine space $Y = \mathbb{A}_k^d$. It follows from Greenberg's theorem that we can find, for each positive integer n, a ring formula φ_n over k, such that, for all fields K containing k, the K-rational points of $\mathcal{L}_n(X)_x$ that can be lifted to a K-rational point of $\mathcal{L}(X)_x$, correspond to the tuples satisfying the interpretation of φ_n in K. We define the arithmetic Poincaré series to be

$$P_{arith}(T) = \sum_{n \geq 0} \chi_c([\varphi_n])T^n \ .$$

As was proven in [7], it is rational over $K_0^{mot}(Var_k)[\mathbb{L}^{-1}] \otimes \mathbb{Q}$.

Suppose that there exists an embedded resolution $h : \tilde{Y} \to Y$ for X, defined over k, which satisfies, after a base change to k^{alg}, the conditions of Theorem 1 or 2. We demand that each component of the exceptional locus contains a k-rational point of the strict transform $\tilde{X}$ that does not lie on any other exceptional component, and that the divisors F_1 and F_2 in Theorem 2 are defined over k. The main result of this section is the following theorem.

Theorem 4. *If these conditions are satisfied, the motivic series P_{arith} and P_{geom} are equal in $K_0^{mot}(Var_k) \otimes \mathbb{Q}[[T]]$.*

Proof. We will prove that, for each field K containing k, and each positive integer n, a K-rational point j_n of $\mathcal{L}_n(X)_x$ lifts to an arc on X if and only if it lifts to a K-rational arc ψ on X. Hence, φ_n and the set of quantifier-free equalities and inequalities describing the constructible set $\pi_n\mathcal{L}(X)_x$ define the same class in $K_0(PFF_k)$, which proves Theorem 4.

So suppose j_n lifts to an arc on X, and let η be a K-rational arc on Y, lifting j_n. By definition, $d_X(\eta) > n$. Theorems 1 and 2 not only give a formula for $d_X(\eta)$, but also give you an optimal approximation for η in $\mathcal{L}(X)$. Now it suffices to observe that this approximation can be chosen to be K-rational. □

We recover a particular case of a theorem in [18] :

Corollary 2. *If (X, x) is the germ of a toric surface singularity, P_{arith} and P_{geom} are equal in $K_0^{mot}(Var_k) \otimes \mathbb{Q}[[T]]$.*

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